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The CW-Inequalities for Vectors in l_1

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Let l_1 denote the metric space the elements of which are all sequences $\mathbf{x}=(x_1, x_2, \ldots)$ of real numbers satisfying $\sum_{i=1}^{\infty}|x_i|<\infty$. The distance $d(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}=(y_1, y_2, \ldots)$ is $d(\mathbf{x}, \mathbf{y})=\sum_{i=1}^{\infty}|x_i-y_i|$. Let $v_0, v_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1}$ be a sequence of $n=m+k$ not necessarily distinct vectors in l_1, where $k>0$, $m=k+2u+1$ and $u\geqslant 0$. We show that

$$\sum_{\substack{0\leqslant i<j<m\\(i-j)(\mathrm{mod}\, m)\in\{\pm 1, \pm 2, \ldots, \pm u\}}} d(v_i, v_j)+\sum_{\substack{0\leqslant i<m\\0\leqslant j<k}} d(v_i, u_j)\geqslant\sum_{0\leqslant i<j<m} d(v_i, v_j)+\sum_{0\leqslant i<j<k} d(u_i, u_j).$$

This solves a conjecture of Deza and Laurent and implies, as special cases, inequalities of Deza, Laurent, Kelly, Barahona and Mahjoub.

1. The CW-Inequalities

Let l_1 denote, as usual, the metric space the elements of which are all real sequences $\mathbf{x}=(x_1, x_2, \ldots)$ satisfying $\sum_{i=1}^{\infty}|x_i|<\infty$. The distance $d(\mathbf{x}, \mathbf{y})$ between the vector $\mathbf{x}$ above and $\mathbf{y}=(y_1, y_2, \ldots)$ is $d(\mathbf{x}, \mathbf{y})=\sum_{i=1}^{\infty}|x_i-y_i|$. In this note we prove the following class of inequalities, conjectured by Deza and Laurent [4].

Theorem 1.1 (*the CW-Inequalities*). *Let $k>0$ and $u\geqslant 0$ be two integers. Put $m=k+2u+1$ and let $v_0, v_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1}$ be a sequence of $m+k$ not necessarily distinct vectors in l_1. Then, for $U=\{\pm 1, \pm 2, \ldots, \pm u\}$,*

$$\sum_{\substack{0\leqslant i<j<m\\(i-j)(\mathrm{mod}\, m)\in U}} d(v_i, v_j)+\sum_{\substack{0\leqslant i<m\\0\leqslant j<k}} d(v_i, u_j)\geqslant\sum_{0\leqslant i<j<m} d(v_i, v_j)+\sum_{0\leqslant i<j<k} d(u_i, u_j). \quad (1.1)$$

In order to state (1.1) in a more compact form, it is convenient to introduce the following notation. For an undirected finite graph $G=(V, E)$, the set of vertices V of which is a set of vectors in l_1, put $d(G)=\sum_{uv\in E} d(u, v)$. Let $W=W(v_0, \ldots, v_{m-1}, u)$ denote the graph the vertices of which are $v_0, v_1, \ldots, v_{m-1}$, in which v_i, v_j is an edge if there is an s, $-u\leqslant s\leqslant u$ such that $i-j\equiv s(\mathrm{mod}\, m)$. Thus W is simply the Cayley graph of the group Z_m with respect to the generating set $1, 2, \ldots, u$. Let $B=B(v_0, \ldots, v_{m-1}, u_0, \ldots, u_{k-1})$ denote the complete bipartite graph on the classes of vertices $\{v_0, \ldots, v_{m-1}\}$ and $\{u_0, \ldots, u_{k-1}\}$, and let K_v and K_u denote the complete graphs on the sets $\{v_0, \ldots, v_{m-1}\}$ and $\{u_0, \ldots, u_{k-1}\}$, respectively. In this notation, inequality (1.1) is simply the assertion $d(W)+d(B)\geqslant d(K_v)+d(K_u)$. As the graphs K_v and K_u are cliques, and the graph W is sometimes called an anti-web, the authors of [4, 5] suggested that the inequalities (1.1) should be called the Clique-Web Inequalities or, for short, the CW-Inequalities.

Several special cases of the CW-Inequalities have been proved by various authors. The case $u=0$ is the class of hypermetric inequalities, first proved by Deza in 1960 [3], and later independently, by Kelly [6]. (Note that if $u=0$ and $k=1$ the obtained inequality is just the triangle inequality.) The case $k=2$ was proved by Barahona and Mahjoub [1]. Deza and Laurent [4, 5] established the cases $u=1$ and $u=2$ as well as the case $k\geqslant(u-1)(u^2+u-2)-2u-1$.

The CW-inequality, when applied to distinct points, defines a facet of the complete cut cone, as proved in [5]. This means, roughly that the inequality does not follow from

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other valid inequalities that do not contain a scalar product of the given one. For more details, including the exact definition of the facets of the complete cut cone, a motivation for the study of these facets as a possible approach to the study of the max-cut problem, and several facets that arise from certain CW-Inequalities with repeated points, see [4, 5].

As stated in Theorem 1.1, the CW-Inequalities hold even when the vectors are not all distinct. Therefore, even if we fix the number of distinct vectors we may obtain infinitely many valid inequalities for them by applying inequality (1.1) for sequences of vectors obtained by repeating each of the given ones as many times as we wish.

2. The Metric Space l_1 and Cut Semi-metrics

It is well known that in order to prove an inequality which is linear in the distances between vectors v_i in l_1 it suffices to prove the same inequality for scalars $\tilde{v}_i \in \{0, 1\}$, for which the distance is simply $d(\tilde{v}_i, \tilde{v}_j) = |\tilde{v}_i - \tilde{v}_j|$. Since this fact is simple, we include it here, for the sake of completeness. For an integer $k \geq 1$, let l_1^k denote the metric space of all real sequences of length k $\mathbf{x} = (x_1, \ldots, x_k)$, where the distance between $\mathbf{x}$ and $\mathbf{y} = (y_1, \ldots, y_k)$ is $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^k |x_i - y_i|$. In particular, l_1^1 is isometric to the space of real numbers with the usual distance function $d(x, y) = |x - y|$. Observe that each l_1^k is isometrically embeddable in l_1, and hence every inequality that holds for vectors in l_1 holds for vectors in l_1^k (and hence also for real numbers). A *cut-semi-metric space* is a pair (V, d) such that V is a set of elements and $d: V \times V \to \{0, 1\}$ is a semi-metric defined by $d(v_i, v_j) = |c(v_i) - c(v_j)|$, where $c: V \to \{0, 1\}$ is an arbitrary function.

Let $A = A(v_1, \ldots, v_n) = \sum_{1 \leq i < j \leq n} a_{ij} d(v_i, v_j)$ be an arbitrary linear combination of the distances between n vectors $v_1, \ldots, v_n$ in l_1. Suppose one wants to prove that $A(v_1, \ldots, v_n) \leq 0$ for all $v_1, \ldots, v_n \in l_1$. Then it suffices to prove it for vectors in l_1^k, for all $k \geq 0$. Indeed, the result for l_1^k implies the validity of the inequality when we replace each $v_i \in l_1$ by the vector of its first k co-ordinates, and by letting k tend to infinity the desired inequality for l_1 follows. Moreover, it suffices to prove the inequality for vectors in l_1^1, i.e. for real numbers. This is because if $v_i = (v_{i1}, \ldots, v_{ik}) \in l_1^k$ $(1 \leq i \leq n)$, then $A(v_1, \ldots, v_n) = \sum_{j=1}^k A(v_{1j}, \ldots, v_{nj})$ and hence the inequality for vectors in l_1^k follows from a summation of the k corresponding inequalities for real numbers. Finally, we claim that it suffices to prove the inequality for the case where each of the real numbers is either 0 or 1. To see this, let $v_1, v_2, \ldots, v_n$ be n reals. By renumbering, if necessary, we may assume that $v_1 \geq v_2 \geq \cdots \geq v_n$. For $1 \leq i < n$, define $c_i: \{v_1, \ldots, v_n\} \to \{0, 1\}$ by $c_i(v_1) = \cdots = c_i(v_i) = 1$ and $c_i(v_{i+1}) \cdots = c_i(v_n) = 0$. Clearly, for every $1 \leq i, j \leq n$, $|v_i - v_j| = \sum_{k=1}^{n-1} (v_k - v_{k+1}) |c_k(v_i) - c_k((v_j)|$. Therefore, by linearity $A(v_1, \ldots, v_n) = \sum_{k=1}^{n-1} (v_k - v_{k+1}) A(c_k(v_1), \ldots, c_k(v_n))$. Since $v_k - v_{k+1} \geq 0$ for all k, the validity of all the inequalities $A(c_i(v_i), \ldots, c_i(v_n)) \leq 0$ for the 0, 1 variables $c_i(v_1), \ldots, c_i(v_n)$ implies the validity of the inequality $A(v_1, \ldots, v_n) \leq 0$.

In view of the above discussion, Theorem 1.1 can be reformulated. Let $k > 0$ and $u \geq 0$ be two integers and suppose $m = k + 2u + 1$. Let $W = W(v_0, \ldots, v_{m-1}, u) = (V, E)$ be the graph defined as follows:

$$V = \{v_0, \ldots, v_{m-1}\} \text{ and } v_i v_j \in E \text{ iff there is an } s, \quad -u \leq s \leq u, \text{ such that } i - j \equiv s \pmod{m}. \tag{2.1}$$

Let $U = \{u_0, \ldots, u_{k-1}\}$ be a set, $U \cap V = \emptyset$. In this notation, Theorem 1.1 is equivalent to the following statement.

PROPOSITION 2.1. *For every function $c: V \cup U \to \{0, 1\}$ the inequality*

$$\sum_{v_i v_j \in E} |c(v_i) - c(v_j)| + \sum_{\substack{1 \le i < m \\ 1 \le j < k}} |c(v_i) - c(u_j)| \ge \sum_{1 \le i < j < m} |c(v_i) - c(v_j)| + \sum_{1 \le i < j < k} |c(u_1) - c(u_j)| \quad (2.2)$$

holds.

3. A COMBINATORIAL LEMMA

For $u \ge 0$, $m \ge 2u + 1$ and for a cyclic permutation $(v_0, v_1, \ldots, v_{m-1})$ of a set of m vertices let $W = W(v_0, \ldots, v_{m-1}, u) = (V, E)$ be the graph defined by (2.1). The following lemma shows that W has an impressive connectivity property. A similar lemma has been proved, independently, by D. Macchi [7].

LEMMA 3.1. *Let S and T be two disjoint subsets of vertices of $W = W(v_0, \ldots, v_{m-1}, u)$ where $|S| \le u$ and $|T| = 2u + 1 - |S|$. Then there are $|S| \cdot |T|$ pairwise edge-disjoint paths $\{P_{s,t}\}_{s \in S, t \in T}$ in W, where $P_{s,t}$ is a path connecting s and t.*

PROOF. We apply induction on m. If $m = 2u + 1$ then W is a complete graph and the assertion of the lemma is trivial. Suppose this assertion holds for $m - 1$ and let us prove it for m $(m > 2u + 1)$. Given two disjoint subsets S and T of vertices of $W(v_0, \ldots, v_{m-1}, n)$ satisfying the hypothesis of the lemma, $|S \cup T| = 2u + 1$ and hence there is a vertex of the graph that does not belong to $S \cup T$. By renumbering, if necessary, we may assume that this vertex is v_0. Consider the graph $W' = W(v_1, \ldots, v_{m-1}, u)$. By the induction hypothesis, in this graph there are $|S|\,|T|$ edge-disjoint paths $\{P'_{st}\}_{s \in S, t \in T}$, where $P'_{s,t}$ is a path in W' connecting s and t. The only edges of W' which are not edges of W are edges of the form $v_{-i}v_{u+1-i}$, where $1 \le i \le u$ and the indices are reduced modulo m. These edges form a matching in W'. Replace each such an edge $v_{-i}v_{u+1-i}$ which appears in a path P'_{st} by the two edges $v_{-i}v_0$ and v_0v_{u+1-i} to obtain a path $P_{s,t}$ in W. Since the edges $v_{-i}v_{u+1-i}$ form a matching, the paths P_{st} obtained in this manner are still pairwise edge-disjoint. This completes the induction and proves the lemma. □

COROLLARY 3.2. *For $u \ge 0$ and $m \ge 2u + 1$ let $c: V \to \{0, 1\}$ be an arbitrary function from the set of vertices of $W = W(v_0, \ldots, v_{m-1}, u) = (V, E)$ to $\{0, 1\}$. Put $r = |\{v_i \in V: c(v_i) = 0\}|$. Then*:
(i) *if $r \le u$ then $\sum_{v_i v_j \in E} |c(v_i) - c(v_j)| \ge r \cdot (2u + 1 - r)$;*
(ii) *if $u \le r \le m/2$ then $\sum_{v_i v_j \in E} |c(v_i) - c(v_j)| \ge u(u + 1)$.*

Both estimates are best possible for all possible values of u, m and r.

PROOF. (i) Put $S = \{v_i \in V: c(v_i) = 0\}$ and let T be an arbitrary subset of cardinality $2u + 1 - |S| = 2u + 1 - r$ of $V \backslash S$. By Lemma 3.1 there are $|S|\,|T| = r(2u + 1 - r)$ edge-disjoint paths $\{P_{s,t}\}_{s \in S, t \in T}$ in W, where $P_{s,t}$ connects s and t. Since $c(s) = 0$ and $c(t) = 1$ there is at least one edge v_iv_j on $P_{s,t}$ such that $|c(v_i) - c(v_j)| = 1$. However, as all the paths are edge-disjoint altogether there are at least $r(2u + 1 - r)$ such edges of W and hence

$$\sum_{v_i v_j \in E} |c(v_i) - c(v_j)| \ge r(2u + 1 - r).$$

(ii) Let S be an arbitrary subset of cardinality u of the set $\{v_i \in V: c(v_i) = 0\}$ and let T be an arbitrary subset of cardinality $u + 1$ of the set $\{v_j \in V: c(v_j) = 1\}$. The $u(u + 1)$

edge-disjoint paths $\{P_{s,t}\}_{s\in S, t\in T}$ guaranteed by Lemma 3.1 supply, as before, the estimate

$$\sum_{v_iv_j\in E} |c(v_i)-c(v_j)| \geqslant u(u+1).$$

To see that both estimates are sharp observe that for all admissible u, m and r, the function $c\colon V\to\{0,1\}$ defined by $c(v_0)=c(v_1)=\cdots=c(v_{r-1})=0$ and $c(v_r)=c(v_{r+1})=\cdots=c(v_{m-1})=1$ satisfies the statements in the corollary with equality. □

REMARK 3.3. Lemma 3.1 and Corollary 3.2 are, in fact, equivalent by Menger's Theorem, or by the max-flow min-cut Theorem (cf., e.g., [2]). For our purposes here, the formulation of Corollary 3.2 is more convenient, but that of Lemma 3.1 may be useful in some other applications.

4. THE PROOF OF THE CW-INEQUALITIES

We can now prove Proposition 2.1, which implies Theorem 1.1, as shown in Section 2. Let $V=\{v_0,\ldots,v_{m-1}\}$ and $U=\{u_0,\ldots,u_{k-1}\}$ be disjoint sets, where $m=k+2u+1$, $k>0$ and $u\geqslant 0$, let $W=W(v_0,\ldots,v_{m-1},u)=(V,E)$ be the graph defined in (2.1) and let $c\colon V\cup U\to\{0,1\}$ be an arbitrary function. We must prove inequality (2.2). Put $x=|\{i\colon 0\leqslant i<m, c(v_i)=0\}|$ and $y=|\{j\colon 0\leqslant j<k, c(u_j)=0\}|$. Clearly we may assume that $x\leqslant m/2$, since otherwise we can replace the coloring c by the coloring $c'=1-c$.

Define $d=\sum_{v_iv_j\in E}|c(v_i)-c(v_j)|$ and

$$D=d+\sum_{\substack{0\leqslant i<m\\0\leqslant j<k}}|c(v_i)-c(u_j)|-\sum_{0\leqslant i<j<m}|c(v_i)-c(v_j)|-\sum_{0\leqslant i<j<k}|c(u_i)-c(u_j)|.$$

Thus, inequality (2.2) is just the statement $D\geqslant 0$. By the definition of x and y, and since $m=k+2u+1$,

$$\begin{aligned}D&=d+x(k-y)+(m-x)y-x(m-x)-y(k-y)\\&=d+x(k-y)+(k+2u+1-x)y-x(k+2u+1-x)-y(k-y)\\&=d+(x-y)^2-(2u+1)(x-y)=d-(x-y)(2u+1-(x-y)).\end{aligned}$$

Consider two possible cases.

CASE 1: $x\leqslant u$. In this case $z=x-y\leqslant x\leqslant u$. The function $f(z)=z(2u+1-z)$ is increasing for $z\leqslant u$ and hence $(x-y)(2u+1-(x-y))\leqslant x(2u+1-x)$. However, by Corollary 3.2 part (i), $d\geqslant x(2u+1-x)$ and hence $D\geqslant 0$, completing the proof in this case.

CASE 2: $x\geqslant u$. Recall that $x\leqslant m/2$. Therefore, by Corollary 3.2, part (ii), $d\geqslant u(u+1)$. On the other hand, the maximum value of the quadratic function $f(z)=z(2u+1-z)$ over all *integers* z is $u(u+1)$ (which is attained at $z=u$ and $z=u+1$). Since $x-y$ is an integer $(x-y)(2u+1-(x-y))\leqslant u(u+1)$ and hence in this case $D=d-(x-y)(2u+1-(x-y))\geqslant u(u+1)-u(u+1)\geqslant 0$, as needed. This completes the proof of Proposition 2.1 and hence that of Theorem 1.1. □

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